Generalised Contractive Mapping

Dr. Komejwar Digambar Dattoba

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At.Post.Limbaganesh,Tq.Dist.Beed
Pin-431126 (Maharashtra) Cell:07588057695,09850203295
harshwardhanpubli@gmail.com, vidyawarta@gmail.com
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PREFACE

The present book eintitled “Generalised contractive Mapping” is the output of my research work in the field of fixed point theory in Hilbert space.

The book consits of five chapters. Chapter – I is introductory in nature which consists of two parts. First part deals with the basic concepts of fundamental fixed point theorems and some useful fixed point results needed for our investigations. Part second of this chapter is of the development of fixed point theory in Hilbert space.

In Chapter – II, a fixed point theorem for semi-generalised ν-contraction mapping in Hilbert space using Parallelogram Law has been established.

In Chapter – III, a fixed point theorem for Kannan type mapping in Hilbert space using Ishikawa same process of Ishikawa iteration scheme has been established. we have extended the above theorem for a pair of mapping and obtained a common fixed point for them.

In Chapter – IV, we have introduced a new definition of generalized contraction mapping in H. space.

Definition : Generalised Contraction

Let C be a closed convex subset of a Hilbert space H. A mapping

$T : C \rightarrow C$ is said to be generalised contraction if for all x, y $\in$ C,

$$||Tx - Ty||^2 \leq a_1||x - y||^2 + a_2||x - Tx||^2 + a_3||y - Ty||^2$$
$$+a_4||x - Ty||^2 + a_5||y - Tx||^2 + a_6||(I - T)x - (I - T)y||^2..$$

Where $a_i \geq 0, \quad \sum_{1}^{6} a_i < 1$

Based upon this definition some fixed point theorems using Ishikawa Iteration scheme have been established. Examples are also provided in support of our investigations.

In chapter – V, we have introduced “Pathak type mapping” in Hilbert space and obtained fixed point theorems by using Mann iteration process. Also we have put an example in support of our fixed point theorem.

The references are given at the end of the book and they are arranged in the alphabetical order. The reference bracket [5] means the 5th reference given at the end of the book.

CHAPETER – I

Generalised contraction mapping in Hilbert space

INTRODUCTION

This chapter is introductory in nature which contains two parts. Part first includes the basic concepts and fixed point theorems which are needed for our investigations. Part two deals with a historical development of the fixed point theorey.

Part – I

1.1 SOME BASIC CONCEPTS

1.1.1 METRIC SPACES

Definition :

Let X be a non-empty set and d be a function from X x X into R^+ such that for all x, y and z in X we have

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric or a distance function and the pair (x, d) is called a metric space. The space (X, d) is also denoted by X if the metric d is understood.

d(x, y) is called the distance between x an y.

Definition :

Let (X, d) be a given metric space.

Let x_0 ε X and real number r > 0 be given.

Then the sets

(i) B $(x_0 ; r)$ = {x ε X /d (x, x_0) > r} is called an open sphere.

(ii) $\overline{B}$ $(x_0 ; r)$ = {x ε X /d $(x, x_0) \leq r$} is called a closed sphere.

(iii) S $(x_0 ; r)$ = {x ε X /d (x, x_0) = r} is called a sphere with the centre at x_0 and radius r.

Convergent Sequence

Definition

Let (X, d) be a metric space and $\{x_n\}$ is said to be convergent if there exists a point x in X such that for each $\varepsilon > 0$ there exists a positive integer N such that for all $n \geq N$

$$d(x_n, x) < \varepsilon$$

i.e. $d(x_n, x) \to 0$ as $n \to \infty$

The point x is called the limit of the sequence $\{x_n\}$ and we write

$x_n \to x$ in the form $\lim_{n\to\infty} = x$

we say that $\{x_n\}$ is a convergent sequence with limit x.

Definition :

Let (X, d) be a metric space and $\{x_n\}$ be a sequence in it. The sequence $\{x_n\}$ is said to be a cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_m, x_n) < \varepsilon \text{ for all } m, n \geq N$$

Theory :

Every convergent sequence in a metric space is a cauchy sequence but not conversely.

Complete Metric Space

Definition :

A metric space X is said to be complete if every cauchy sequence in X converges to a point in X.

Subsequence

Definition :

A subsequence of a sequence $\{x_n\}$ is a sequence whose terms are chosen from the terms of the sequence $\{x_n\}$ and arranged in the same order as their relative order in $\{x_n\}$. A subsequence of $\{x_n\}$ is often designated as $\{x_{n_i}\}$ with term $x_{n_1}, x_{n_2}, \ldots$

Note : if a sequence $\{x_n\}$ converges to x then any subsequence of $\{x_n\}$ also converges to x.

Bounded Sequence

Definition :

A sequence $\{x_n\}$ is said to be bounded if there exists numbers m_1, m_2 such that

$m_1 \leq \{x_n\} \leq m_2$ for $n \in N$

Monotonic sequence

Definition :

A sequence $\{x_n\}$ is said to be strictly monotonically increasing if

$x_{n+1} > x_n$, for all n

A sequence which is either monotonically increasing or decreasing in called monotonic sequence.

Theorem :

A monotonic sequence is convergent if only if it is bounded.

Compact Metric space

Definition :

A metric space X is said to be compact if every sequence in X has convergent subsequence.

Open Set

Definition :

Let (X, d) be a metric space. A subset M of X is said to be open if and only if to each $x \in M$, there exists $r > 0$, such $S(x, r) \subset M$.

Closed Set

Definition :

A subset M of a metric space (X, d) is said to be closed if the complement of M in X is open.

1.1.2 NORMED LINEAR SAPCES

Let X be a real or complex vector space or linear space of finite or infinite dimension. Let K be the field of complex number C. or real numbers R.

Norm of a Space

Definition :

Let X be a linear space on K. A norm on X is a real function $||.|| : X \to R^+$ defined on X such that for any $x, y \in X$, and for $\lambda \in K$, we have

(i) $||x|| \geq 0$

(ii) $||x|| = 0$ if and only if $x = 0$

(iii) $||\lambda x|| = |\lambda| \; ||x||$

(iv) $||x+y|| \leq ||x|| + ||y||$

Definition :

A normed linear space is a vector with a norm and it is denoted by $(X, ||.||)$

Result :

A norm on X defines a metric d on X which is given by

$d(x, y) = ||x - y||$ for all $x, y \in X$

and is called the metric induced by the norm. Thus every normed linear space X is a metric space with this metric defined on X.

Theorem :

In a normed linear space, nor n in continuous function i.e. if $x_n \to x$ then $||x_n|| \to ||x||$

Strong Conversion

Definition :

A sequence $\{x_n\}$ in a normed linear space (or normed space). X is said to be strongly convergent (or convergent in the norm) if there exists an $x \in X$ such that

$\lim_{n\to\infty} ||x_n - x|| =$

0 i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \to x$

we say that $\{x_n\}$ converges strongly to x and x is the strong limit of $\{x_n\}$

1.1.3 Banach Spaces

A normed Linear space X is said to be complete it every cauchy sequence in X converges to an element of X. A complete normed linear space is called a Banach Space.

Convex Set

Definition :

Let B be an arbitrary Banach Space. A convex set in B is a non-empty sub-set S with the property that for all x, y

$\in$ X, $\bar{z} = tx + (1 - t)y \in S$ for every real number t such that $0 \le t \le 1$.

Note :

(i) The empty set and the set containing one point are convex.

(ii) Every subspace of a vector space is convex. In particular, every vector space is convex.

1.1.4 HILBERT SPACE

Inner Product Space

Definition :

Let X be a linear space over the scalar field K (Real or complex). A function < , > : X x X $\rightarrow$ K is called an inner product on X if for all x, y, z $\in$ X and $\lambda \in$ K we have,

(i) $< x. x > \ge 0$ and $< x, x > = 0$ if and if $x = 0$

(ii) $< x, y >$ - $< \overline{y, x} >$ where bar denotes complex conjugates.

(iii) $<\lambda x, y> = \lambda < x, y >$

(iv) $<x + y, z> = < x, z > + < y, z >$

An inner product space or pre Hilbert space is a linear space X with the inner product on it.

If K = R, then <x,y> is a real number and if

K = C, then it a complex number.

Properties of inner Product Space :

(i) Every inner product space is a normed space but not all normed space are inner product spaces.

(ii) In a inner product space, the inner product is continuos.

(iii) If x and y are in an inner product space, then

$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$

which is known as parallelogram law.

(iv) If x, y are in an inner product space,

then $|\langle x, y\rangle| \leq ||x||.||y||$

(v) Let X be a normed linner space in which the Parallelogram Law holds. Then X can be made into an inner product space by defining the given norm as $\langle x, x\rangle = ||x||^2$

Hilbert Space

Definition :

A complete inner product space is called a Hilbert Space.

Result :

Every Hilbert Space is a Banach space but converse is not true.

Fixed Point

Definition :

Let X be a set and $T : X \rightarrow X$ be a self map.

A fixed point of T is a point $x \in X$

Such that $Tx = x$ i.e. the image Tx coincides with x.

Example,

(i) A mapping $x \rightarrow x^3$ of R into itself has three fixed points (0, -1, 1) .

(ii) A translation has no fixed point.

(iii) A rotation of the plane has a single fixed point i.e. the centre of rotation.

(iv) A mapping $Tx = x^2 - 6$ defined on R has $x = -2$, $x = 3$ are fixed points.

The following definitions in Hilbert Space are due to Browder and Petryshyn [7]

Let C be a convex subset of a real Hilbert Space H and T be a nonlinear (possibly) mapping from C into H, then we have.

Definition :

T is said to be strictly contractive if there exists a constant k with 0< K <1 such that

$||Tx-Ty|| < K ||x-y||$ for all $x, y \in C$

Definition :

T is said to be contractive (or for all $x, y \in C$),

$$||Tx - Ty|| < ||x - y||$$

Definition :

T is said to be strictly pseudo contractive if there exists a constant $0 < K < 1$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + K \, ||(I - T)\, x - (I - T)y||^2$$

for all x, y, $\in$ C

Definition :

T is said to be pseudo contractive if for all x, y, $\in$ C.

$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2$

These mappings admit iterative methods for the construction of their fixed points.

Identity Mapping

Definition :

The identify mapping $I : X \rightarrow X$ defined by $I(x) = x$

Continuous Mapping

Definition :

A mapping T of a metric space X into a metric space Y is said to be continuous at x $\in$ X.

of $x_n \rightarrow x$ in X then $Tx_n \rightarrow Tx$ in Y

Definition :

Let H be a Hilbert Space and C be a convex subset of H. T be a mapping from C in to H.

A mapping T is said to be monotone [1] if

Re <Tx – Ty ; x – y > ≥ 0 for all x, y in C

Now we mention here some fixed point theorems.

1.1.5 FUNDAMENTAL FIXED POINT THEOREMS

Brouwer's [8] and Schayder's [41] fixed point theorems are fundamental theorems in the field of fixed point theory and its applications. Though Brouwer obtained his result in 1912. Poincare proved a slightly different version of it in 1986 which was subsequently rediscovered by Bohl P. in 1904.

Brower's Fixed point Theorem

Every continuous map of the closed unit ball S = {x / ||x|| < 1} in R^n, the n-dimensional Euclidean space to itself has a fixed point.

Birkhoff and Kellog [3] were the first to prove fixad point theorems in infinite dimensional spaces. They considered continuous self-maps defined on compact subsets of C[0, 1] and L^2 [0, 1] and established the existence of fixed points for them Schauder [41] generalized these results.

Schuder's Fixed Point Theorem

Let C be a non-empty convex compact subset of normed linner space X. Then every continous self map of C has a fixed point.

Many authors have extended Schauder's theorem in different spaces. Tychonoff [43] extended Brouwer's result to a compact convex subset of a locally convex linear topological space.

1.1.5 ITERATIVE METHODS

Mann Iteration Process :

Mann [30] gave the following iteration process. For a self-mapping T of a closed bounded interval of the real line having a unique fixed point, the interation process

$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n + Tx_n$ ……. (1.1.7)

with

$\alpha_n = \frac{1}{n+1}$, converges to the fixed point of T as $n \to \infty$

Ishikawa Iteration Process

In 1974, Ishikawa [19] introduced the following iterative procedure.

Let C be a non-empty convex subset of a Hilbert space H and T be a self map on C. then the iteration scheme $\{x_n\}_{n=0}^{\infty}$ introduced by Ishikawa is a follows.

For any $x_0 \in C$,

$$\left.\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n\, Tx_n, \quad n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\, Ty_n, \quad n \geq 0 \end{aligned}\right] \quad ..(1.1.8)$$

Where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers which satisfy the following three conditions.

$$\left.\begin{aligned} &\text{(i)} && 0 \leq \alpha_n \leq \beta_n < 1 \\ &\text{(ii)} && \lim_{n\to\infty} \beta_n = 0 \\ &\text{(iii)} && \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \end{aligned}\right] \quad ...(1.1.9)$$

Ishikawa Identity

For any x, y, z in a Hilbert space H and a real number t,

$$\|tx + (1-t)y - z\|^2 = t\|x-y\|^2 + (1-t)\|y-z\|^2 - t(1-t)\|x-y\|^2 \quad ...(1.1.10)$$

Part – II

1.2 Historical Developments of a Fixed Point Theory in Hilbert Spaces

The theory of Hilbert spaces is originated in the year 1912 with the work 'Grundzuge einer allegemeinon Theorie der linearen Integralqleichungen' of the great German mathematician D. Hilbert [18]. However, several years elapsed before an axiomatic basic was provided by the famous mathematician J. Von Neumann [34]. The modern developments in Hilbert spaces are concerned largely with the theory of operator on the spaces.

Browder initiated the study of fixed point theory of non-expansive mappings in Hilbert spaces without compactness conditions. In 1965, Browder [5] proved the following theorem.

Theorem :

Let Br be a closed ball of radius $r > 0$ in real or complex Hilbert space H, ∂ Br be the boundary Brand S be a nonlinear contraction map of Br into H such that $Sx - \lambda x \neq 0$

for all x in Br and any $\lambda \neq 1$. Using the theory of monotone operators developed in [31, 4] Browder [5] showed that T has atleast one fixed point in Br.

Petryshyn [36] studied an iteration method for the actual construction of fixed points of a nonlinear contraction map S under the additional assumption that S is demicompact. He has proved his main result in the following way.

Theorem :

Let S be a demicompact contraction of Br into H such that $Sx-\lambda x \neq 0$, for all $x \in \partial Br$ and $\lambda > 1$, then the set of fixed points Fr of S lying in Br is a nonempty convex set and for any $x_0 \in Br$ and any $\beta > 0$ such that $0 < \beta < 1$ the sequence $\{x_{n+1}\}$ determined by the process

$x_{n+1} = \beta\, r_n\, Sx_n + (1-\beta)\, x_n$, $n = 0, 1, 2, ..$ where the real numbers r_n, n= 0, 1, 2, ... are given by

$$r_n = \begin{cases} 1 \text{ if } ||Sx_n|| \leq r \\ \frac{r}{||Sx_n||} \text{ if } ||Sx_n|| \geq r \end{cases}$$

converges to a fixed point $\bar{z} \in F_r \subset B_r$ of S

Browder and Petryshyn [7] introduced the four classes of nonlinear mappings (Strictly contractive, contractive, strictly Pseudo contractive and Pseudo contractive) which admit iterative methods for the construction of their fixed points. They established the following basic existence result.

Theorem :

Let C be a closed bounded convex subset of the Hilbert space H. T be a contractive mapping of C into C. Then T has at least one fixed point in C.

Based upon this theorem a number of theorems have been proved by the authors. We give here few of them,

Theorem :

It T is contractive (non-expansive) mapping of C into C, where C is a closed convex subset of a Hilbert space H and the set F(T) of fixed points of T in C is non-empty, then the mapping defined by $T_\lambda = \lambda I + (1 - \lambda) T$ for any given λ with $0 < \lambda < 1$ is a reasonable wanderer from C into C with the same fixed points as T.

Corollary :

If T is contractive (non-expansive) mapping of C into C with non-empty set F (T) of fixed points of T in C and it the mapping defined by

$T_\lambda = \lambda I + (1 - \lambda) T$ for a given λ with $0 < \lambda < 1$, then T_λ maps with C into C, T_λ has the same fixed points as T and T_λ is a symptotically regular.

Theorem :

Let T be a self-map of a bounded closed convex subset C of a Hilbert space H. Suppose T is contractive and demicompact. Then the set F(T) of fixed points of T in C is a non-empty convex set and for any given $x_0 \in C$ and any fixed $\lambda > 0$ with $0 < \lambda < 1$, the sequence $\{x_n\} = \{T^n_\lambda x_0\}$ determined by the sequence

$$x_n = \lambda T_{x_{n-1}} + (1 - \lambda) x_{n-1} \ : n = 1, 2, 3, \ldots$$

converges strongly to a fixed point of T in C.

Hicks and Huffman [17] generalized theorem (1.1.4) and (1.2.6) in generalized Hilbert space (see theorem 6.7 of [17]).

Ishikawa [19] has introduced a new interation scheme, called as Ishikawa interation scheme and proved that a sequence of Ishikawa iterates for a Lipschizian pseudocontractive mapping in a convex compact subset of a Hilbert space converges strongly to a fixed point of this mapping.

Das and Debata [13] have extended and generalized the result of Ishikawa [19] by taking simultaneously a more generalized iteration scheme involving a family of maps and secondly by taking less restrictive hemicontractive mappings. Their result states as follows :

Theorem :

Let $\{T_j\}$, $j = 1, 2, \ldots. K$, $K \geq 2$ be a family of hemicontractive maps defined on a convex, compact subset C of a Hilbert space H and have at least one common fixed point in C. Let the family of maps $\{T_j\}$ satisfy

$$||T_i x - T_j y|| \leq M ||x - y||$$

for all x, y $\in$ C and pairs (i, j), M being a positive constant. Then the sequence $\{x_n\}$ converges to a common fixed point of the family of maps $\{T_j\}$ where x_n is defined iteratively for each positive integer n by $x_1 \in C$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_k u_{k-1} (n)$$

Were,

$$u_o (n) = x_n.\ u_j (n) = (1 - \beta_n) x_n + \beta_n T_j u_{j-1} (n)$$

for j = 1, 2, ... K and $\{\alpha_n\}$, $\{ \beta_n)$ are real sequence in [0, 1] such that

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$ for n = = 1,2,...

(ii) $\lim_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n^{k-1} = \infty$ for each $K \geq 2$

The authors [13] claimed that for K = 2, $T_1 = T_2$, the above theorem includes the result of in Ishikawa [19] as a corollary. They further claim that the Ishikawa iteration can be extended to Lipschitzain hemi contractive mappings.

In 1976 and 1983, Rhoades [38] and Naipally and Singh [33] studied the Ishikawa iteration scheme, respectively, and put forth the following questions ;

Can the Ishikawa iteration procedure be extended to quasi-contractive and hemi contractive mappings?

Liu Qihou [27, 28] studied the above questions and proved the convergence theorem of the sequence of Ishikawa iterates for quasi contractive mappings and Lipschitzain hemi contractive mappings. After this, Liu Qihou [29] continue to study the above questions and proved the following two theorems.

Theorem :

Let C be a convex compact subset of a Hilbert space and T : C → C, a continuous hemi contractive mapping. Suppose that the number of the fixed points of T is finite. Then, for each $x_0 \in C$, the sequence of Ishikawa iterates $\{x_n\}_{n=0}^{\infty}$ must converge to a fixed point of T.

Theorem :

Let C be a convex compact subset of Hilbert space and T : C → C, a continuous generalized contractive mapping. Then, for each $x_0 \in C$, the sequence of Ishikawa iterates $\{x_n\}_{n=0}^{\infty}$ must converage to a fixed point of T.

Here we complete the brief survey of the development of the fixed point theory in Hilbert Space.

CHAPETER – II

Fixed point theorem & identity mapping

2.1 Let B denote a Banach space with the norm $||.||$ and C be a closed subset of B. the transformation F : C $\rightarrow$ C is called contraction if there exists a constant K with $0 < K < 1$ such that $||Fx - Fy|| \leq K||x-y||$. If K = 1 it is called non-expansive. Banach contraction principle states that a contraction mapping C into C has a unique fixed point. This conclusion is also true for $||Fx - Fy||^2 \leq K\, ||x - y||^2$ but it is no longer true for K = 1. However, Browder [5] has proved that every non-expansive mapping of a closed, bounded and convex subset of a uniformly, convex Banach space has at least one fixed point.

In 1971, Goebel and Zlotkiewicz [15] have proved the following theorem :

2.1.1 Theorem :

If C is a closed and convex subset of B and F : C $\rightarrow$ C statisfies

(i) $F^2 = I$, I is identity mapping

(ii) $||Fx - Fy|| \leq K||x - y||$ where $0 \leq K < 2$,

then F has at least one fixed point.

In 1991, Sharma and Sahu [42] have obtained the most generalised theorem from the result of Goebel an Zlotkiewicz [15]

2.1.2 Theorem :

Let F be a mapping of a Banach space X into itself and satisfy

(i) $F^2 = I$, Where I : Identify mapping

(ii) $$||Fx - Fy|| \leq \frac{a||x-Fx|| \; ||y-Fy||}{||x-Fy|| + ||y-Fy|| + ||x-y||} + \frac{b||x-Fx|| \; ||y-Fy||}{||x-y||} + c\{||x-Fx|| + ||y-Fy\} + d\{||x-Fy|| + ||y-Fx||\} + e||x-y||$$

For every x, y $\in$ X and $x \neq y$, a, b, c, d, $e \geq 0$ $a + 4b + 4c + 4d + e < 2$, $2d + e < 1$. Then F has a unique Fixed Point.

2.1.3 Semi – generalised ν– contraction mapping [25]

definition :

A mapping T from a closed subset C into C of a Hilbert Space H satisfying

$$||Tx - Ty||^2 \leq \alpha ||x - Tx||^2 + \beta||y - Ty||^2 + \nu ||x - y||^2$$

Is called semi-generalised ν – contraction with

$0 < \alpha + \beta + \nu < 1$ and $\alpha\beta\nu > 0$

2.1.4 Parallelogram Law [26] P. 130

Let H be a Hilbert space and x, y $\in$ H Then

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + ||y||^2 \quad \ldots\ldots \text{(i)}$$

From (i) we have.,

$$||x + y||^2 \leq 2\left[||x||^2 + ||y||^2\right] \quad \ldots\ldots \text{(ii)}$$

$$||x + y||^2 \leq 2\left[||x||^2 + ||y||^2\right]$$

Now we recall the definition of Identity mapping

2.1.5 A mapping I : X → X is called identity mapping if I(x) = x for all x in X

Using (2.1.4 and 2.1.5) and the equation (ii) we prove the following theorem

2.1.6 Theorem :

Let F be a mapping of a Hilbert space H into itself satisfying

(i) $F^2 = I$: I is the identity mapping

(ii) $||Fx - Fy||^2 < \alpha ||x - Fx||^2 + \beta||y - Fy||^2 + \nu||x - y||^2$

For all x, y in H where $\alpha, \beta, \nu \geq 0$ and $0 < \alpha + \beta + \nu < 1$

Proof : Let x be a fixed point of H.

Set $y = \frac{1}{2}(F + I)x$, $z = F(y)$

and $u = 2y - z$

Then by using (i) and (ii) we get

$$||z - x||^2 = ||F(y) - x||^2 = ||F(y) - F^2(x)||^2$$
$$\leq \alpha||F(x) - F^2(x)||^2 + \beta||y - F(y)||^2 + \nu||y - F(x)||^2 \leq$$
$$\alpha||F(x) - x||^2 + \beta||y - Fy||^2$$

$$+\nu\left|\ 2||y-F(y)||^2+2||F(y)-F(x)||^2\right.$$

by 2.1.4

Next,

$$||F(y)-F(x)||^2 \le 2||F(y)-x||^2+2||x-F(x)||^2$$

$$\le 4||x-F(x)||^2 \quad \text{proved} \quad ||F(y)-x|| \le ||x-F(x)||$$

$$\therefore \quad ||z-x||^2 \le (\alpha+8\nu)||x-F(x)||^2 + +(\beta+2\nu)||y-F(y)||^2$$

Now,

$$||u-x||^2 = ||2y-F(y)-x||^2$$

$$= ||(F+I)\,x-F\,(y)-x||^2$$

$$= ||F(y)\ -\ F(x)||^2$$

$$< \alpha||x-F(x)||^2 + \beta||y-F(y)||^2 + \nu\,||x-y||^2$$

$$< \alpha||x - Fx||^2 + \beta||y - Fy||^2 +$$

$$\nu\left[2||x - Fy||^2 + 2||y - Fy||^2\right]$$

But

$$||x - Fy||^2 \leq 2||x - Fx||^2 + 2||Fx - Fy||^2$$

$$\leq 2||x - Fx||^2 + 8||x - F||^2$$

$$= 10||x - F(x)||^2$$

$$\therefore ||u - x||^2 \leq \alpha||x - Fx||^2 + \beta||y - Fy||^2$$

$$+\nu\left[20||x - Fx||^2 + 2||y - Fy||^2\right]$$

$$\therefore ||u - x||^2 \leq (\alpha + 20\nu)||x - Fx||^2 + (\beta + 2\nu)||y - Fy||^2$$

Hence,

$$||z - u||^2 \leq 2||z - x||^2 + 2||u - x||^2$$

$$\leq 2(\alpha + 20\nu)||x - Fx||^2 + 2(\beta + 2\nu)||y - Fy||^2$$

$$+2(\alpha + 20\nu)||x - Fx||^2 + 2(\beta + 2\nu)||y - Fy||^2$$

$$\therefore ||z - u||^2 \leq 4(\alpha + 20\nu)||x - Fx||^2 + 4(\beta +$$

$$2\nu)||y - Fy||^2$$

$$\therefore \nu < 1$$

We have,

$$||z-u||^2 = ||u-z||^2 = ||2y-z-z||^2$$

$$= 4||y-z||^2$$

$$= 4||y-F(y)||^2$$

Hence,

$$4||y-Fy||^2 = ||z-u||^2$$

$$\leq 4(\alpha+20\nu)||x-Fx||^2 + 4(\beta+$$

$$2\nu)||y-Fy||^2$$

$$\therefore ||y-Fy||^2$$

$$\leq (\alpha+20\nu)||x-Fx||^2 + (\beta+2\nu)||y$$

$$-Fy||^2$$

$$|1-(\beta+2\nu)|\ \|y-Fy\|^2 \leq (\alpha+2\nu)\ \|x-Fx\|^2$$

$$\|y-Fy\|^2 \frac{\alpha+20\nu}{1-(\beta+2\nu)}||x-Fx||^2$$

Let $G = \frac{1}{2}(F+I)$, then for all $x \in H$

$$||G^2x-Gx||^2 = ||Gy-y||^2$$

$$= ||\frac{1}{2}(F+I)y-y||^2$$

$$= \frac{1}{4}(y-Fy)||^2$$

$$\leq \frac{1}{4} \frac{\alpha+20\nu}{|1-(\beta+2\nu)|} ||x - Fx||^2$$

$$= \frac{1}{4} \frac{\alpha+20\nu}{|1-(\beta+2\nu)|} ||x - (2Gx - x)||^2$$

$$= \frac{\alpha+20\nu}{|1-(\beta+2\nu)|} ||Gx - x||^2$$

$$\therefore \ ||G^2x - Gx|| \leq ||Gx - x|| \qquad \because \frac{\alpha+20\nu}{1-(\beta+2\nu)} < 1$$

Therefore the sequence $\{x_n\}$ defined by $x_n = G^n(x)$ is a Cauchy sequence in H.

Since H is complete

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} G^n x = x_o$$

Consider,

$$||x_o - Gx_o||^2 = ||x_o - x_{n+1} + x_{n+1} - Gx_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + 2||x_{n+1} - Gx_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + 2||Gx_n - Gx_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + 2||\frac{1}{2}(F+I)x_n - \frac{1}{2}(F+I)x_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + \frac{1}{2}||(F+I)x_n - (F+I)x_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + ||(x_n - x_o)||^2 + ||Fx_n + Fx_o||^2$$

$$\leq 2||x_o - x_{n+1}||^2 + ||(x_n - x_o)||^2 + \alpha||x_n + Fx_n||^2$$

$$+\beta||x_o - Fx_o||^2 + \nu||x_n - x_o||^2$$

$$= 2||x_o - x_{n+1}||^2 + (1+\nu)||x_n - x_o||^2 +$$

$$\alpha||x_{n+1} - x_n|| + \beta||Gx_o - x_o||^2$$

As $n \rightarrow \infty$, we get

$$||x_o + Gx_o||^2 \leq \beta||Gx_o - X_o||^2$$

$\Rightarrow Gx_o = x_o$ $\quad\because \beta < 1$

Now,

$$x_0 = Gx_o$$

$$= \frac{1}{2}(F+1)x_o$$

$$2x_0 = F(x_o) + Ix_o$$

$$2x_0 = F(x_o) + x_o$$

$\Rightarrow F(x_o) = x_o$

$\Rightarrow x_o$ is a fixed point of F

<u>Uniqueness</u> :

Let if possible x_o and y_o are two distinct fixed points of F in H

Then,

$$||x_o - y_o||^2 = ||Fx_o - Fy_o||^2$$

$$\leq \alpha||x_o - y_o||^2 + \beta||y_o - Fy_o||^2 + \nu||x_o - y_o||^2$$

$$\therefore ||x_o - y_o||^2 \leq \nu||x_o - y_o||^2$$

$$\Rightarrow (1-\nu)||x_o - y_o||^2 \leq 0$$

Since $\nu < 1$ and $||.|| \nless 0$

We have $||x_o - y_o||^2 = 0$

$$\Rightarrow ||x_o - y_o||^2 = 0$$

$$\Rightarrow x_o = y_o$$

Hence x_0 is a unque fixed point F.

CHAPTER – III

Kannan Type Mapping in H. Space

In this chapter we have used Ishikawa iteration for process to obtain a fixed point for a Kannan type mapping Hilbert space. Further, it is shown that the theorem can be extended for two different mappings T_1 and T_2 by the process of Ishikawa [19].

3.0 Kannan mapping in Hilbert space [23]

Definition :

Let C be a closed subset of a Hilbert space H and T : $C \rightarrow C$ be a self map satisfy the condition.

$$||Tx - Ty|| \leq [||x - Tx|| + ||y - Ty||]$$

For all x, y $\in$ C where $0 < \alpha < \frac{1}{2}$. Then T is called Kannan mapping in Hilbert space.

3.0.1 In 1991 Kannan type mapping in Hilbert [24] space was defined by Koparde and Waghmode and proved the following theorem by using the Picard's iteration process.

Definition :

A mapping $T : C \to C$, Where C is a subset of a Hilbert space H, is called a Kannan type mapping if

$$||Tx - Ty||^2 \leq \alpha\left[||x - Tx||^2\right] + ||y - Ty||^2$$

For all x, y $\in$ C and $0 < \alpha < \frac{1}{2}$

3.0.2 **<u>Theorem</u> :** [24] Let C be a closed subset of a Hilbert space H. Let T be a self mapping on C satisfying

$$||Tx - Ty||^2 < \alpha\left[||x - Tx||^2\right] + ||y - Ty||^2$$

For all x, y $\in$ C and $0 < \alpha < \frac{1}{2}$, Then T has a unique fixed point in C.

For our work we need the definition (3.0.1) and the theorem (3.0.2) and Ishikawa iteration process (I – 1.1.8 and I – 1.1.9)

Our result runs as follows :

3.0.3 Theorem :

Let C be a closed convex subset of a Hilbert space H. Let T be a self map on C satisfying (3.0.1) with $\alpha(1+\beta_n^2) < 1$. Suppose x_0 is any point in C and the sequence $\{x_n\}$ associated with T is defined by Ishikawa scheme I –1.1.8 and I –1.1.9. Suppose that $\{\alpha_n\}$ is bounded away from zero.

i.e. $\underline{\lim}\, \alpha_n = \alpha > 0$. If the sequence $\{x_n\}$ converges to P, then P is a unquie fixed point of T.

Proof : Equation I –1.1.8 implies that

$$x_{n+1} - x_n = \alpha_n(Ty_n - x_n)$$

Suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ and since $\{\alpha_n\}$ is bounded away from zero we have

$$||Ty_n - x_n||^2 \to 0$$

………. (3.0.4)

using triangle inequality, we have

$$||Ty_n - P||^2 \leq \{Ty_n - x_n|| + ||x_n - x_{n+1}||\}^2$$

$$\to 0 \text{ as } n \to \infty$$

i.e. $||Ty_n - P||^2 \to 0$

using I –1.1.8 and I –1.1.10, where t stand for β_n we obtain the following inequality :

$$||y_n - Ty_n||^2 = ||\beta_n Tx_n + (1-\beta_n)x_n - Ty_n||^2$$

$$= \beta_n ||Tx_n - Ty_n||^2 + (1-\beta_n)||x_n - Ty_n||^2 - \beta_n(1-\beta_n)||Tx_n - x_n||^2$$

$$= \beta_n ||Tx_n - Ty_n||^2 - \beta_n(1-\beta_n)||Tx_n - x_n||_2^2$$

by 3.0.4

….. 3.0.6

since T satisfies that

$$||Tx - Ty||^2 \leq \alpha\{||x - Tx||^2 + ||y - Ty||^2\}$$

we have,

$$||Tx_n - Ty_n||^2 \leq \alpha\left\{\left\{||x_n - Tx_n||^2 + ||y_n - Ty_n||^2\right\}\right.$$

Using 3.0.6 , the above inequality becomes

$$||Tx_n - Ty_n||^2 \leq \alpha\left\{||x - Tx_n||^2 + \beta_n||Tx_n - Ty_n||^2 - \beta_n(1-\beta_n)||Tx_n - x_n||^2\right.$$

$$\Rightarrow ||Tx_n - Ty_n||^2 \leq \alpha\{[1-\beta_n(1-\beta_n)]||x_n - Tx_n||^2 + \beta_n||Tx_n - Ty_n||^2$$

$$\Rightarrow (1-\alpha\beta_n)||Tx_n - Ty_n||^2 \leq \alpha(1-\beta_n+\beta_n^2)||x_n - Tx_n||^2$$

$$\Rightarrow ||Tx_n - Ty_n||^2 \leq \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}||x_n - Tx_n||^2 \quad \ldots(3.0.7)$$

Now we use triangle inequality to get

$$||x_n - Tx_n||^2 \leq \left\{||Tx_n - Ty_n|| \quad + ||Ty_n - x_n||\right\}^2$$

Therefore 3.0.7 becomes

$$||Tx_n - Ty_n||^2 \leq \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}\left[||Tx_n - Ty_n||^2 + ||Ty_n - x_n||^2 + 2||Tx_n - Ty_n||.||Ty_n - x_n||\right]$$

$$= \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}||Tx_n - Ty_n||^2$$

by 3.0.4

$$\Rightarrow \left[1 - \frac{\alpha(1-\beta_n+\beta_n^2)}{1-\alpha\beta_n}\right] \; ||Tx_n - Ty_n||^2 \leq 0$$

$$\Rightarrow \; [1 - \alpha(1+\beta_n^2)]||Tx_n - Ty_n||^2 \leq 0$$

Since $(1+\beta_n^2) < 1 \; for \; 0 < \; \alpha < \frac{1}{2}$ and

$||.|| \nless 0$: we have

$||Tx_n - Ty_n||^2 = 0$ i.e. $||Tx_n - Ty_n|| \to 0$

….. (3.0.8)

Hence,

$$||x_n - Tx_n||^2 \leq \{||x_n - Ty_n|| + ||Tx_n - Ty_n||\}^2 \to 0$$

.....(3.0.9)

and

$$||P - Tx_n||^2 \leq \{||P - x_n|| + ||x_n - Tx_n||\}^2 \to 0$$

.....(3.1.0)

Now we show that P is a fixed point of T.

As T satisfies the inequality in the statement we have,

$$||Tx_n - TP||^2 \leq \alpha \, ||x_n - Tx_n|| + ||P - TP||^2$$

$$\to \alpha ||P - TP||^2$$

by 3.0.9

(..... 3.1.1)

Next, using triangle inequality

$$||P-TP||^2 \leq \{||P-Tx_n|| + ||Tx_n - TP||\}^2$$

$$\leq \alpha||P-TP||^2 \quad \text{by} \quad (3.1.0 \text{ and } 3.1.1)$$

$$\Rightarrow (1-\alpha)||P-TP||^2 \leq 0$$

Since $0 < \alpha < \frac{1}{2}$ and $||.|| \nless 0$, we have

$$||P-TP||^2 = 0 \quad \text{i.e.} \; ||P-TP|| = 0$$

TP = P   i.e. P is a fixed point of T

Let if possible P and q be two fixed points of T, then

$$||P-q||^2 = ||TP-Tq||^2$$

$$\leq \alpha \left\{||P-TP||^2 + ||q-Tq||\right\}^2$$

$$||P-q|| = 0 \qquad P = q$$

Therefore a mapping $T : C \to C$ has a unique fixed point in C.

We verify the above theorem by the following example.

Example :

Let T : [0, 1] → [0, 1] be a mapping defined by

$Tx = \frac{x}{4}$, for all x in [0, 1]

Then 0 is the only fixed point of T

Now,

$$||Tx - Ty||^2 = ||\frac{x}{4} - \frac{y}{4}||^2$$

$$= \frac{1}{16}||x - y||^2$$

And

$$\alpha\left[||x - Tx||^2 + ||y - Ty||^2\right]$$

$$= \alpha\left[||x - \frac{x}{4}||^2 + ||y - \frac{y}{4}||^2\right]$$

$$= \frac{9\alpha}{16}\left[||x||^2 + ||y||^2\right]$$

There for $\frac{1}{9} < \alpha < \frac{1}{2}$ we have

$$||Tx - Ty||^2 \leq \alpha \left| \; ||x - Tx||^2 + ||y - Ty||^2 \right.$$

Also for any $x_o \in [0, 1]$

$x_n = \frac{1}{4^n} x_o$ → 0 as → n

By the statement of the theorem this limit 0 is the unique fixed point of T. This verifies the theorem.

3.0.2 Theorem :

Let C be a closed convex subset of a Hilbert space H. Let T_1 and T_2 be two self mapping on C satisfying

$$||T_1x - T_2y||^2 \leq \alpha\left[||x - T_1x||^2 ||y - T_2||^2\right]$$

$\forall$ x, y in C with $0 < \alpha < \frac{1}{2}$ and $a + \beta_n^2 < 1$

Suppose x_0 is any point in C and the sequence $\{x_n\}$ associated with T_1 and T_2 defined by Ishikawa scheme I –1.1.8 and I –1.1.9 Suppose further that $\{\alpha_n\}$ is bounded away from zero. If the sequence $\{x_n\}$ converges to P, then P is a unique common fixed point of T_1 and T_2 .

Proof :

From Chapter I –1.1.8 we have,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_2 y_n$$

$$\Rightarrow x_{n+1} - x_n = \alpha_n(T_2y_n - x_n)$$

Suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ and $\{\alpha_n\}$ is a sequence bounded away from zero we have $||T_2 y_n - x_n||^2 \to 0$

....(3.2.1)

Using triangle inequality it follows that

$$||T_2 y_n - P||^2 \le \{||T_2 y_n - x_n|| + ||x_n - x_{n+1}||\}^2$$

$$\to 0 \text{ as } n \to \infty$$

i.e. $||T_2 y_n - P||^2 \to 0$ ....(3.2.2)

using (I – 1 .1.8 and 1.1.10), where t stands for β_n we obtain the following inequality

$$||y_n - T_2 y_n||^2 = ||\beta_n T_1 x_n + (1 - \beta_n) x_n - T_2 y_n||^2$$

$$= \beta_n ||T_1 x_n - T_2 y_n||^2 + (1 - \beta_n)||x_n - T_2 y_n||^2$$

$$- \beta_n (1 - \beta_n) ||T_1 x_n - x_n||^2$$

..... (3.2.3)

Using the inequality of the statement we have,

$$||T_1x_n - T_2y_n||^2$$

$$\leq \alpha\left[||x_n - T_1x_n||^2 + ||y_n - T_2y_n||^2\right]$$

$$\leq \alpha\left[||x_n - T_1x_n||^2 + \right.$$

$$\left.\beta_n||T_1x_n - T_2y_n||^2\right]$$

$$+(1-\beta_n)\,||x_n - T_2y_n||^2 -$$

$$\beta_n\,(1-\beta_n)$$

$$||T_1x_n - x_n||^2$$

by … 3.2.3

$$\Rightarrow ||T_1x_n - T_2y_n||^2 \leq \alpha\,\beta_n||T_1x_n - T_2y_n||^2$$

$$[\alpha - \alpha\,\beta_n\,(1-\ \beta_n]\ ||x_n - T_1x_n||^2$$

$$\Rightarrow (1-\alpha\,\beta_n)||T_1x_n - T_2y_n||^2$$

$$\leq [\alpha - \alpha\,\beta_n(1-\beta_n)]||x_n - T_1x_n||^2$$

$$\Rightarrow ||T_1x_n - T_2y_n||^2 \leq \frac{\alpha - \alpha\beta_n + \alpha\,\beta_n^2}{1-\alpha\,\beta_n}||x_n - T_1y_n||^2$$

…. (3.2.4)

Now,

$$||x_n - T_1x_n||^2$$

$$\leq \{||T_1x_n - T_2y_n|| + ||T_2y_n - x_n||\}^2$$

Therefore 3.2.4 becomes

$$\Rightarrow ||T_1x_n - T_2y_n||^2$$

$$\leq \frac{\alpha - \alpha\beta_n + \alpha\beta_n^2}{1 - \alpha\,\beta_n}\Big[||T_1x_n - T_2y_n||^2$$

$$+ ||T_2y_n - x_n||^2 + 2||T_1x_n - T_2y_n||x||T_2y_n -$$

$$x_n||\Big]$$

Using Triangle inequality

$$\Rightarrow \left[1 - \frac{\alpha - \alpha\beta_n + \alpha\,\beta_n^2}{1 - \alpha\,\beta_n}\right]||T_1x_n - T_2y_n||^2 \leq 0$$

by 3.2.1

$$\Rightarrow [1 - \alpha\,(1 + \beta_n^2)]||T_1x_n - T_2y_n||^2 \leq 0$$

Since $0 \leq \alpha + \alpha\,\beta_n^2 < 1$ for $0 < \alpha < \frac{1}{2}$,

we have $||T_1x_n - T_2y_n||^2 = 0$

…. (3.2.5)

Next, using triangle inequality, we have

$$||x_n - T_1x_n||^2$$

$$\leq \{||x_n = T_2y_n|| + ||T_1x_n - T_2y_n||\}^2 \qquad \text{.... (3.2.6)}$$

$\rightarrow 0$ by 3.2.1 and 3.2.5

and

$$||P - T_1x_n||^2 \leq \{||P - x_n|| + ||x_n - T_1y_n||\}^2$$

$\rightarrow 0$ as $x_n \rightarrow P$ and 3.2.6

Now we try to show that P is a fixed point of both T_1 and T_2 :

consider,

$$||T_1x_n - T_2P||^2$$

$$\leq \alpha\left[||x_n - T_1x_n||^2 + ||P - T_2P||^2\right] \qquad \text{.... (2.2.7)}$$

$\rightarrow \alpha||P - T_2||^2$ by 3.2.6

Using triangle inequality we have

$$||P - T_2P||^2 \leq \{||P - T_1x_n|| + ||T_1x_n - T_2P||\}^2$$

$\leq \alpha||P - T_2P||^2$ by 3.2.7

$\Rightarrow (1-\alpha)||P - T_2P||^2 \leq 0$

Since $0 < \alpha < \frac{1}{2}$ and $||.|| \nless 0$ we have,

$||P - T_2P|| = 0$ $\qquad$ $T_2P = P$

Similarly we can show that P is also a fixed point of T_1 i.e. $T_1P = P$. Thus P is a common fixed point of T_1 and T_2.

Let if possible P and q be two common fixed points of T_1 and T_2, then

$$||P - q||^2 = ||T_1P - T_2q||^2$$

$$\leq \alpha\left[||P - T_1P||^2 + ||q - T_2P||^2\right]$$

$$= 0$$

$$\Rightarrow ||P - q|| = 0$$

$$\Rightarrow P = q$$

Hence T_1 and T_2 have unique common fixed point in C.

CHAPTER – IV

Generalised contraction mapping in Hilbert space

4.0.0 Introduction

The well-known Banach [2] contraction principle has been extended by a number of research workers working in the field of fixed point theory in several directions to different spaces which can be stated as follows

Let X be a banach space and C be a closed convex subset of X, then a contraction mapping T of C into itself satisfying.

$$||Tx - Ty|| \leq ||x - y||$$

for some $\alpha \in (0, 1)$ and for all x, y in C has a unique point $P \in C$ such that TP = P.

the definition of contraction mapping has undergone successive generalisations [39] in complete metric space by R. Kannan [21], Reich [40], Hardy and Rogers [16] proved some fixed point theorem by considering the following general form of contraction mapping.

Let X by a complete metric space, then a contraction mapping T of C into itself satisfying.

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Ty) + a_3 d(y, Tx) + a_4 d(x, Tx) + a_5 d(y, Ty)$$

where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i < 1$

Khan and Imdad [22] considered the above generalised contraction in Banach space in the following form :

T be a self map of closed convex subset of a Banach space X satisfying

$$||Tx - Ty|| \leq a||x - y|| + b[||x - Tx|| + ||y - Ty] + C[||x - Ty|| + ||y - Tx]$$

for every x and y in C, a, b, c $\geq$ 0 and $0 \leq a + 4b + 4c < 2$

Naimpally and Singh [33] used the two contraction conditions and proved some fixed point theorems.

Ganguly [14] in his recent paper defined a generalised non-expansive mapping in the following way :

A self map T of a subset of a normed linear space X is said to be genralised non-expansive if

$$||Tx - Ty|| < \max\{||x - y||, ||x - Tx||, ||y - Ty||, ||x - Ty||, ||y - Tx||\}$$

The purpose of This Chapter :

By considering the above generalisations of contraction mapping in different spaces, we have introduced the following new definition of generalised contraction mapping in Hilbert space.

Our definition runs as follows

4.1.0 Generalised contraction mapping Definition :

Definition :

Let C be a closed convex subset of a Hilbert space H. A mapping $T : C \rightarrow C$ is said to be generalised contraction if for all $x, y \in C$

$$||Tx - Ty||^2$$

$$\leq a_1||x - y||^2 + a_2||x - Tx||^2$$

$$+ a_3||y - Ty||^2$$

$$+a_4||x - Ty||^2 + a_5||y - Tx||^2 + a_6||(I - T)x -$$

$$(I - T)y||^2$$

…. 4.0.2

where $a_i \geq 0$ and $\sum_{i=1}^{6} a_i$

$= 1$ ... 4.0.3

we observe that

(i) If $a_2 = a_3 = a_4 = a_5 = a_6 = 0$, $0 < \sqrt{a_1} = K < 1$

We get T strictly contractive mapping.

(ii) If we put $\sqrt{a_1} = 1$, $a_2 = a_3 = a_4 = a_5 = a_6 = 0$

we obtain T as non-expansive mapping

(iii) If we put $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ and $a_6 < 1$, we obtain T as stricly pseudo-contractive.

(iv) If we put $a_1 = a_6 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ we obtain T as a pseudo-contractive mapping.

(v) If we put $a_2 = a_3$ and $a_1 = a_4 = a_5 = a_6 = 0$ T becomes a Kannan type mapping which we have studied in Chapter– III.

Our first result runs as follows :

4.0.3 <u>Theorem</u> :

Let C be a closed convex subset of a real Hilbert space H. Let T : C→ C such that it satisfies 4.0.2 and 4.0.3 with $a_4 \neq a_6$ and $0 < a_3 + a_4 + a_6 < 1$ Further we assume that T is montone. Suppose x_o is any point in C and the sequence $\{x_n\}$ associated with T is defined by Ishikawa scheme I –1.1.8 and I –1.1.9. Suppose $\underline{\lim}\, \alpha_n = \alpha > 0$. If the sequence $\{x_n\}$ converges to P, then P is a fixed point of T

Proof :

From equation I – 1.1.8 we have

$$x_{n+1} - x_n = \alpha_n(Ty_n - x_n)$$

Suppose $x_n \to P$, then $||x_{n+1} - x_n||^2 \to 0$ and

Since $\{\alpha_n\}$ is bounded away from zero,

$$||Ty_n - x_n||^2 \to 0$$

…. (A)

Using triangle inequality, we have ;

$$||Ty_n - P||^2 \leq \left|||Ty_n - x_n|| + ||x_n - x_{n+1}||\right|^2$$

$$\to 0 \text{ as } n \to \infty$$

Using I – 1 .1.8 and I – 1.1 .10. where t stands for β_n we obtain the following :

$$||y_n - x_n||^2 = ||\beta_n Tx_n + (1-\beta_n)x_n - x_n||^2$$

$$= \beta_n||Tx_n - x_n||^2 - \beta_n(1 - \beta_n)||Tx_n - x_n||^2$$

$$= \beta_n^2||Tx_n - x_n||^2$$

$$\leq ||Tx_n - x_n||^2$$

$$\leq \{||Tx_n - Ty_n|| + ||Ty_n - x_n||\}^2$$

Using Triangle inequality,

$$\leq ||Tx_n - Ty_n||^2 + ||Ty_n - x_n||^2$$

$$+ 2||Tx_n - Ty_n|| x ||Ty_n - x_n||$$

….. (4.0.4)

Now,

$$||y_n - Tx_n||^2 = ||\beta_n Tx_n + (1 - \beta_n) x_n - Tx_n||^2$$

$$= (1 - \beta_n)||x_n - Tx_n||^2 - \beta_n (1 -$$

$$\beta_n)||Tx_n - x_n||^2$$

$$= (1 - \beta_n)^2 ||Tx_n - x_n||^2$$

$$\leq ||Tx_n - x_n||^2$$

$$\leq \left| ||Tx_n - Ty_n|| + ||Ty_n - x_n||^2 \right|$$

Using T. inequality

$$\leq ||Tx_n - Ty_n||^2 + ||Ty_n - x_n||^2 +$$

$$2||Ty_n - Tx||\ x$$

$$||Ty_n - x_n||$$

….(4.0.5)

Since T satisfies 4.0.2, we have

$$||Tx_n - Ty_n||^2$$

$$\leq a_1||x_n - y_n||^2 + a_2\ ||x_n - Tx_n||^2 +$$

$$a_3||y_n - Ty_n||^2 + a_4||x_n - Ty_n||^2 +$$

$$a_5||y_n - Tx_n||^2 + a_6\Big\{||x_n - y_n||^2$$

$$||Tx_n - Ty_n||^2 + 2 < x_n - y_n,\ \ Tx_n - Ty_n >\Big\}$$

$$\leq (a_1 + a_6)||x_n - y_n||^2 + a_2||x_n - Tx_n||^2$$

$$+a_3||y_n - Ty_n||^2 + a_4||x_n - Ty_n||^2 +$$

$$a_5||y_n - Tx_n||^2 + a_6||Tx_n - Ty_n||^2$$

…… (4.0.6)

Since H is a real Hilbert Space and T is monotone

Using relations 4.0.4 and 4.0.5 in 4.0.6. we get

$$||Tx_n - Ty_n||^2$$

$$\leq (a_1 + a_6)\Big[||Tx_n - Ty_n||^2 + ||Ty_n$$

$$- x_n||^2$$

$$+2||Tx_n - Ty_n|| \ \times$$

$$||Ty_n - x_n||\Big]$$

$+a_2[x_n - Ty_n||^2 + ||Tx_n - Ty_n||^2 +$

$2||x_n - Ty_n||$

$\times ||Tx_n - Ty_n ||] + a_3[||x_n - y_n||^2 +$

$||x_n - Ty_n||^2 +$

$2||x_n - y_n||x||x_n - Ty_n||] + a_4||x_n -$

$Ty_n||^2$

$+a_5[||Tx_n - Ty_n||^2 + ||Ty_n - x_n||^2 +$

$2||Ty_n - Tx_n|| \times$

$||Ty_n - x_n||] + a_6||Tx_n - Ty_n||^2$

$\Rightarrow 1 - (a_1 + a_2 + a_3 + a_5 + 2a_6)||Tx_n - Ty_n||^2$

$\leq (a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6)||Ty_n - x_n||^2$

$+2(a_1 + a_2 + 2a_3 + a_5 + a_6)||Ty_n - Tx_n|| \times$

$||Ty_n - x_n||$

$\Rightarrow (a_4 - a_6)||Tx_n - Ty_n||^2 \leq (1 + a_3)||Ty_n - x_n||^2$

$+2(a_1 + a_2 + 2a_3 + a_5 + a_6)||Ty_n - Tx_n|| \times$

$||Ty_n - Tx_n||$

$$\text{since} \sum_{i=1}^{6} a_i \le 1$$

Taking limit of above as $n \to \infty$

We have,

$||Tx_n - Ty_n||^2 \to 0$ $\therefore a_4 \neq$

a_6 and by (A)

… (4.0.7)

Using triangle inequality, we have

$||Tx_n - Ty_n||^2 \le [||x_n - Ty_n|| + ||Tx_n - Ty_n||]^2$

$\to 0$ by (A) and 4.0.7 as $n \to \infty$

…. (4.0.8)

And

$||P - Tx_n||^2 \le \left|||P - x_n|| + ||x_n - Tx_n\right|^2$

$\to 0$ as $x_n \to P$ and 4.0.8

….. (4.0.9)

Now we show that P is a fixed-point of T. Since T satisfies 4.0.2, we have

$||Tx_n - TP||^2 \le a_1 ||x_n - P||^2 + a_2 ||x_n - Tx_n||^2$

$$a_3\,||P-TP||^2 + a_4\,||x_n - TP||^2 +$$

$$a_5\,||P-Tx_n||^2 + a_6\left\{||x_n - P||^2 +\right.$$

$$||Tx_n - TP||^2$$

$$-2 < x_n - P,\ Tx_n - TP\}$$

$$\rightarrow \frac{a_3 + a_4}{1 - a_6}\,||P$$

$- TP||^2$ since by data and 4.0.8 and 4.0.9 .....4.1.0

Now, using triangle inequality

$$\Rightarrow ||P-TP||^2 \leq \{||P-Tx_n|| + ||Tx_n - TP||\}^2$$

$\Rightarrow ||P-TP||^2 \leq \frac{a_3+a_4}{1-a_6}\,||P-TP||^2$ by 4.0.9

And 4.1.0

$$\Rightarrow \left[1 - \frac{a_3 + a_4}{1 - a_6}\right]||P-TP||^2 \leq 0$$

$$\Rightarrow [1-(a_3 + a_5 + a_6)]\,||P-TP||^2 \leq 0$$

$$\Rightarrow ||P-TP||^2 \leq 0$$

by data

$$||P-TP|| = 0$$

$$||.|| \nless 0$$

$\Rightarrow TP = P$

i.e. P is a fixed pint of T

This proves the theorem.

Now we have generalised the theorem 4.0.3 as follows

4.2.0 <u>Theorem</u> :

Let C be a closed convex subset of a real Hilbert space H. Let T_1 and T_2 be two self maps satisfying 4.0.2 and 4.0.3 with $a_4 \neq a_6$ and $0 \leq a_3 + a_4 + a_6 < 1$. Further we assume that T is monotone suppose x_0 is any point in C and $\{x_n\}$ associated with T_1 and T_2 is defined by Ishikawa scheme I – 1.1.8 and I – 1.1.9. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence bounded away from zero. If the sequence $\{x_n\}$ converges to P, then P is a fixed point of both T_1 and T_2

Proof :

Exactly on the same lines, we have proved this theorem as in Chapter [III, see Theorem 3.2.0].

CHAPTER – V

Pathak type Mapping

5.0.0 Introduction

The well-known Banach contraction principle states that every contraction mapping of a complete metric space x into itself has a unique fixed point. This principle has been generalised in various ways by many authors.

This principle has not found much attention in terms of a rational expression. Some authors who have made an attempt to generalise this principle through such an expression are as follows :

In 1975, Dass and Gupta [12] have tried to generalise Banach contraction principle for the mapping T : X → X satisfying

$$d(Tx, Ty) \leq \alpha\, d(y, Ty)\frac{[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x, y) \ldots (i)$$

for all x, y in X and α unique fixed point in X.

in April 1979, H. Chatterjee [9] has obtained fixed point theorems for mapping of Dass and Gupta, by using inequality (i), in arbitrary topological space. Using the same mapping of Dass and Gupta he has obtained in [10] and [11] respectively, a unique fixed point theorem for a pair of continuous self-mapping of a metric space and a fixed point theorem of a continuous mapping of a compact metric space.

In 1980, Jaggi and Dass [20] have proved the following fixed point theorem through a rational expression which comes out to be an extension of the well-known Banach's contraction mapping theorem.

Theorem :

Let F be a self map defined on a metric space (x, d) satisfying the following :

(i) for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$

$$d[F(x), F(y) \leq \frac{\alpha\, d[x, F(x)].d\,[y, F(y)]}{d[x, F(y)] + d[y, F(x)] + d(x, y)}$$

$$+\beta d(x, y),$$

for all x, y ε X and $x \neq y$

(ii) there exists $x_0 \in X$ such that

$$\{F^n(x_0) \supset \{F^n k\,(x_0)\} \text{ with } \lim_{n\to\infty} F^n k\,(x_0) \in X$$

Then F has unique fixed point

$$u = \lim_{n\to\infty} F^n k\,(x_0)$$

in 1982, Sharma and Yue1 [44] proved the following fixed point theorem for mapping in a normed space as follows :

Theorem :

Let T be a mapping of a normed space X into itself satisfying

$$d(Tx, Ty) \le q \max \left\{ d(x, y), \frac{d(y,\ Ty) + [1 + d(x,\ Tx)]}{1 + d\,(x,\ Tx)}, \frac{1}{2}\,\frac{d(x,\ Ty) + [1 + d(x,\ Tx) + d\,(y,\ Ty)]}{1 + d\,(x,\ y)} \right\}$$

for all x, y in X and $0 < q < 1$ where the sequence $\{x_n\}$ is given by

$$x_n + 1 = (1 - C_n)\, x_n + C_n\, Tx_n \text{ for } n \ge 0 \text{ where}$$

C_n satisfy $C_o = 1$, $0 < C_n < 1$ for $n > 0$ and

$\sum C_n$ diverges and $\lim_n C_n = h > 0$

If $\{x_n\}$ converges in X, then it converges to a fixed point of T.

In 1984, Sharma and Yuel [45] have obtained the following unique fixed point theorem.

Theorem :

Let T be a mapping of a complete metric

Space X into itself such that

$$d(Tx, Ty) \leq \alpha \left\{\frac{d(x, Tx)\, d(y, Ty)}{d(x, y)}\right\} + \beta\{d(x, Tx) + d(y, Ty)\} + v\{d(x, Ty) + d(y, Tx)\} + \delta\, d(x, y)$$

for all x, y in X where

$$0 \leq \frac{\beta + v + \delta}{1 - \alpha - \beta - v} < 1, \quad \beta + v < 1, \quad 2v + \delta < 1,$$

$v \geq 0$. Then T has a unique fixed point.

In the same year 1984, Bajaj [1] has obtained the following theorem for pair of mappings :

Theorem :

Let S and T be a pair of self mappings of a complete metric space (x, d) and satisfy the inequality

$$d(Sx, Ty) \leq \alpha \frac{d(x, Sx),\ d(x, Ty) + [d(x, y)]^2 + d(x, Sx),\ d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}$$

for all x, y in X with $x \neq y$, $0 < \alpha < 1$

and $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$

Then S and T have common fixed point. Further if d(x, Sx) + d(x, y) + d(x, Ty) = 0 implies

d(Sx, Ty) = 0, then S and T have a unique common fixed point.

Naik [32] had obtained a unique fixed point theorem as follows :

Theorem :

Let T be a non-empty, bounded closed and convex subset of a reflexive Banach space and let K have normal structure, Let T : K→ K be a continuous map of K satisfy

$$||Tx - Ty|| \leq \frac{||x - Tx||\ ||x - Ty||\ +\ ||y - Ty||\ ||y - Tx||}{||x - Ty|| + ||y - Tx||} \qquad \text{if } ||x - Ty||\ ||y - Tx|| > 0$$

$$= 0 \qquad \text{otherwise}$$

Then T has a unique fixed point in K.

In 1988, H.K. Pathak [35] have defined The following mapping in a normed space using more general rational inequality.

Definition :

Let X be a normed space then T, a self mapping of X called a 'generalised contraction mapping' if.

$$||Tx - Ty|| \leq q \max\left\{||x - y||\frac{||x - Tx||[1 - ||x - Ty||]}{1 + ||x - Tx||}, \frac{||x - Ty||[1 - ||x - Ty||]}{1 + ||x - Ty||}, \frac{||Tx - y||[1 - ||y - Ty||]}{1 + ||Tx - y||} \frac{||y - Ty||[1 - ||Tx - y||]}{1 + ||y - Ty||}\right\}$$

for all x, y in X where $0 < q < 1$.

He proved the following fixed point theorem.

Let X be a closed, convex subset of a normed space N and T be a 'generalised contraction mapping' of X and T be continuous on X. Let $\{x_n\}$ be the sequence of Mann iterates associated with T defined by

$x_{n+1} = (1 - C_n) x_n + C_n T x_n$ for $n \geq 0$ where

C_n satisfy $C_o = 1$, $0 < C_n < 1$ for $n > 0$

and $\lim_{n \to \infty} C_n = h > 0$, *If* $\{x_n\}$ converges in x., then it converges to a fixed point of T.

He has extended the above theorem 1 for pair of mappings T_1 and T_2 and obtained common fixed point for them.

Purpose of This Chapter :

From the above all results it is clear that the Banach contraction principle has undergone many generalisations through rational expressions in metric spaces as well as in normed spaces. But this principle has not found any attention in terms of rational expressions in Hilbert spaces. In this chapter, we have made an attempt to generalise this principle through such rational expressions in Hilbert spaces.

Now we define a Pathak type mapping in Hilbert space as follows :

5.0.1 Definition : A self mapping T : C → C, Where C is a closed convex subset of a Hilbert space H, called Pathak type mapping if it satisfies the condition.

$$||Tx - Ty||^2 \leq q \max\left\{ ||x - y||^2, \frac{||x - Tx||^2\left[1 - ||x - Ty||^2\right]}{1 + ||x - Tx||^2}, \frac{||x - Ty||^2\left[1 - ||x - Tx||^2\right]}{1 + ||x - Ty||^2}, \frac{||Tx - y||^2\left[1 - ||y - Ty||^2\right]}{1 + ||Tx - y||^2}, \frac{||y - Ty||^2\left[1 - ||Tx - y||^2\right]}{1 + ||y - Ty||^2} \right\}$$

for all x, y in C where $0 < q < 1$

<u>Main Result :</u>

We here establish two fixed point theorems using the technique as appeared in Rhoades [37] and Yuel and Sharma [44] for mappings satisfying 5.0.1.

Our first result runs as follows :

5.0.2 Theorem :

Let C be a closed, convex subset of a real Hilbert space H and T be a mapping satisfying 5.0.1and T be continuous on C, $\{x_n\}$ be sequence of Mann iterates associated with mapping T given by

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$ for $n \geq 0$ where $\{\alpha_n\}$ satisfies

(d) $\alpha_o = 1$ $_{=1}$ (de) $0 < \alpha_n < 1$ for $n > 0$

(iii) $\lim_{n\to\infty} \alpha_n = h \in (0,1)$,

If $\{x_n\}$ converges in C, then it converges to a fixed point of T.

Proof :

Let $z \in C$ such that $\lim_{n\to\infty} x_n = z$

Now we show that z is the fixed point of T consider,

$||z - Tz||^2 = ||z - x_{n+1} + x_{n+1} - T_z||^2$

$$\leq ||z - x_{n+1}||^2 + ||x_{n+1} - T_z||^2 +$$

$$2||z - x_{n+1}|| \text{ x } ||x_{n+1} - T_z||$$

$$= ||z - x_{n+1}||^2 + ||(1 - \alpha_n)x_n + \alpha_n Tx_n - Tz||^2$$

$$+ 2||z - x_{n+1}|| |x ||x_{n+1} - Tz||$$

$$= ||z - x_{n+1}||^2 + ||(1 - \alpha_n)x_n - (1 - \alpha_n)Tz +$$

$$\alpha_n Tx_n - \alpha_n Tz||^2 +$$

$$+ 2||z - x_{n+1}|| \times$$

$$||x_{n+1} - Tz||$$

Therefore,

$$||z - Tz||^2 \leq ||z - x_{n+1}||^2 + (1 - \alpha_n)^2||x_n$$

$$- Tz||^2$$

$$+\alpha_n^2||Tx_n - Tz||^2 + 2\alpha_n^2(1 -$$

$$\alpha_n)^2||x_n - Tz|| \text{ x}$$

$$||Tx_n - Tz|| + 2||z - x_{n+1}|| \text{ x } ||x_{n+1} -$$

$$Tz||$$

$$\leq ||z - x_{n+1}||^2 + (1 - \alpha_n)^2 ||x_n - Tz||^2 +$$

$$\alpha_n^2\, q \max \left\{ ||x_n - z||^2, \right.$$

$$\frac{||x_n - Tx_n||^2 \left[1 - ||x_n - Tz||^2\right]}{1 + ||x_n - Tx_n||^2} -$$

$$\frac{||x_n - Tx_n||^2 \left[1 - ||x_n - Tx_n||^2\right]}{1 + ||x_n - Tz||^2}$$

$$\left. \frac{||Tx_n - z||^2 \left[1 - ||z - Tz||^2\right]}{1 + ||Tx_n - z||^2}, \frac{||z - Tz||^2 \left[1 - ||Tx_n - z||^2\right]}{1 + ||z - Tz||^2} \right\}$$

$$+ 2\, \alpha_n^2 (1 - \alpha_n)^2 ||x_n - Tz||\ x\ ||Tx_n - T_z|| +$$

$$2||z - x_{n+1}||\ x\ ||x_{n+1} - T_z||$$

we observe that

$$||x_n - Tx_n||^2 = ||x_n - x_{n+1}||^2 / \alpha_n^2$$

Thus the above inequality reduces to

$$||z - Tz||^2 \leq ||z - x_{n+1}||^2 + (1 - \alpha_n)^2 ||x_n - Tz||^2$$

$$+\alpha_n^2\, q \max\left\{ ||x_n\right.$$

$$-z||^2 \frac{||x_n - x_{n+1}||^2/\alpha_n^2\left[1-||x_n - Tz||^2\right]}{1+||x_n - x_{n+1}||^2/\alpha_n^2}$$

$$\frac{||x_n - Tz||^2\left[1-||x_n - x_{n+1}||^2/\alpha_n^2\right]}{1+||x_n - Tz||^2}$$

$$\left.\frac{||Tx_n - z||^2\left[1-||z - Tz||^2\right]}{1+||Tx_n - z||^2}, \frac{||z - Tz||^2\left[1-||Tx_n - z||^2\right]}{1+||z - Tz||^2}\right\}$$

$$+ 2\,\alpha_n^2(1-\alpha_n)^2||x_n - Tz||\ x\ ||Tx_n - Tz|| + 2||z$$

$$- x_{n+1}\ ||x$$

$$||x_{n+1}$$

$$- Tz||$$

Now taking the limit as a n $\rightarrow \infty$ and

Using (iii) and continuity of T we have

$$||z - Tz||^2 \le (1-h)^2||z - Tz||^2 + h^2\, q \max\ \{0,$$

$$0,, \frac{||z - Tz||^2}{1+||z - Tz||^2}, \frac{||z - Tz||^2\left[1-||z - Tz||^2\right]}{1+||z - Tz||^2},$$

$$\left.\frac{||z-Tz||^2\left[1-||z-Tz||^2\right]}{1+||z-Tz||^2}\right\}$$

$$\Rightarrow ||z-Tz||^2 \le (1-h)||z-Tz||^2 + hq \max\{0,\ 0,$$

$$\frac{||z-Tz||^2}{1+||z-Tz||^2}, \frac{\|z-Tz\|\left[1-||z-Tz||^2\right]}{1+||z-Tz||^2},$$

$$\left.\frac{||z-Tz||^2\left[1-||z-Tz||^2\right]}{1+||z-Tz||^2}\right\}$$

$$\Rightarrow ||z-Tz||^2$$

$$\le (1-h)||z-Tz||^2 + \frac{hq||z-Tz||^2}{1+||z-Tz||^2}$$

$$\Rightarrow (1-1+h)||z-Tz||^2 \le \frac{hq\,||z-Tz||^2}{1+||z-Tz||^2}$$

$$\Rightarrow ||z-Tz||^2 \le q\,\frac{||z-Tz||^2}{1+||z-Tz||^2} \qquad \because h>0$$

$$\Rightarrow ||z-Tz||^2 + ||z-Tz||^4 \le q\,||z-Tz||^2$$

$$||z-Tz||^4 \le -(1-q)||z-Tz||^2$$

Suppose that $z \neq Tz$, then let $\Rightarrow ||z-Tz||^2 = \delta$

then we have $\delta < -(1-q)$

which is a contradiction since $||.|| \nless 0$

Hence Tz = z i.e. z is the fixed point of T

Now we give an example to verify the above theorem.

Example :

Let H = R, the set of real numbers regarded as a Hilbert space.

Let T : [0, 1] → [0, 1] be a mapping such that Tx = x/2

Then,

$$||Tx - Ty||^2 = \frac{1}{4}\ ||x - y||^2$$

Setting $\frac{1}{4} \leq q < 1$ we see that all the conditions of above theorem are satisfied are satisfied. Here 0 is the fixed point of T.

Now, we extended the above theorem 5.0.2 for a pair of mappings T_1 and T_2

5.0.3 <u>Theorem</u> :

Let C be a closed, convex subset of a Hilbert space H and Let T_1 and T_2 be pair of mapping satisfying.

$$||T_1x - T_2y||^2 \leq q \max\left\{ ||x - y||^2, \frac{||x - T_1x||^2\left[1 - ||x - T_2y||^2\right]}{1 + ||x - T_1x||^2}, \frac{||x_n - T_2y||^2\left[1 - ||x - T_1x||^2\right]}{1 + ||x - T_2y||^2}, \frac{||T_1x - y||^2\left[1 - ||y - T_2||^2\right]}{1 + ||T_1x - y||^2}, \frac{||y - T_2y||^2\left[1 - ||T_1x - y||^2\right]}{1 + ||y - T_2y||^2} \right\}$$

for all x, y in C with $0 < q < 1$ and

T_1 and T_2 are continuous on C.

The sequence $\{x_n\}$ of Mann iterates associated with T_1 and T_2 are given by

for $x_o \in C$, Set $x_{2n+1} = (1 - \alpha_n)x_{2n} + \alpha_n T_1 x_2 n$

and $x_{2n+2} = (1 - a_n)x_{2n+1} + \alpha_n T_2 x_{2n+1}$

for $n \geq 0$ where $\{\alpha_n\}$ satisfy conditions

(i) to (iii) of theorem 5.0.2. Here we further assume that $h \neq \frac{1}{2}$. if $\{x_n\}$ converges to z in C and if z is a fixed point of either T_1 or T_2 then z is the common fixed point of T_1 and T_2

Proof :

Let $Z \in C$ such that $\lim_{n\to\infty} x_n = z$

and let $T_1 z = z$

Now we show that z is the common fixed point of T_1 and T_2

consider,

$$||z - T_2 z||^2 = ||z - x_{2n+1} + x_{2n+1} - T_2 z||^2$$

$$\leq ||z - x_{2n+1}||^2 + ||(1-\alpha_n)x_{2n} + \alpha_n T_1 x_{2n} - T_2 z||^2$$

$$+2||z - x_{2n+1}||x \ ||x_{2n+1} - T_2 z||$$

$$\leq ||z - x_{2n+1}||^2 + ||(1-\alpha_n)x_{2n} - (1-\alpha_n)T_2 z + \alpha_n T_1 x_{2n} - \alpha_n T_2 z||^2 + 2||z - x_{2n+1}||x||x_{2n+1} - T_2 z||$$

$$\Rightarrow ||z - T_2 z||^2 \le ||z - x_{2n+1}||^2 + (1-\alpha_n)^2 ||x_{2n} - T_2 z||^2 + \alpha_n^2 ||T_1 x_{2n} - T_2 z||^2 + 2\alpha_n^2 (1-\alpha_n)^2 ||x_{2n} - T_2 z|| \, x \, ||T_1 x_{2n} - T_2 z|| + 2||z - x_{2n+1}|| \, x \, ||x_{2n+1} - T_2 z||$$

$$\Rightarrow ||z - T_2 z||^2 \le ||z - x_{2n+1}||^2 + (1-\alpha_n)^2 ||x_{2n} - T_2 z||^2 + \alpha_n^2 q \max \left\{ ||x_{2n} - z||^2, \frac{||x_{2n} - T_1 x_{2n}||^2 \left[1 - ||x_{2n} - T_2 z||^2\right]}{1 + ||x_{2n -} T_1 x_2 n||^2}, \frac{||x_{2n} - T_2 z||^2 \left[1 - ||x_{2n} - T_1 x_{2n}||^2\right]}{1 + ||x_{2n} - T_2 z||^2} \right.$$

$$\frac{||T_1x_{2n}-z||^2\left[1-||z-T_2z||^2\right]}{1+||T_1x_{2n}-z||^2},\frac{||z-T_2z||^2\left[1-||T_1x_{2n}-z||^2\right]}{1+||z-T_2z||^2}\Bigg\}$$

$$+2\ \alpha_n^2(1-\alpha_n)^2||x_{2n}-T_2z||\times\ ||T_1x_{2n}-T_2z||$$

$$+2||z-x_{n+1}||\ x\ ||x_{2n+1}-T_2z||$$

we observe that

$$||x_{2n}-T_1x_{2n}||^2=||x_{2n}-x_{2n+1}||^2/\alpha_n^2$$

Hence the above inequality reduces to

$$||z-T_2z||^2\leq||z-x_{2n+1}||^2+(1-\alpha_n)^2||x_{2n}-T_2z||^2$$

$$+\alpha_n^2 q\max\Bigg\{||x_{2n}-z||^2,\frac{||x_{2n}-x_{2n+1}||^2/\alpha_n^2\left[1-||x_{2n}-T_2z||^2\right]}{1+||x_{2n-}x_{2x+1}||^2/\alpha_n^2},$$

$$\frac{||x_{2n}-T_2z||^2\left[1-||x_{2n}-x_{2n+1}||^2/\alpha_n^2\right]}{1+||x_{2n}-T_2z||^2},$$

$$\frac{||T_1x_{2n}-z||^2\left[1-||z-T_2z||^2\right]}{1+||T_1x_{2n}-z||^2},$$

$$\frac{||z-T_2z||^2\left[1-||T_1x_{2n}-z||^2\right]}{1+||z-T_2z||^2},$$

$$+2\,\alpha_n^2(1-\alpha_n)^2||x_{2n}-T_2z||x\,||T_1x_{2n}-T_2z||$$

$$+2||z-x_{n+1}||\;x\;||x_{2n+1}-T_2z||$$

Now taking the limit as $n \to \infty$ and using (iii) and continuity of T_1

We obtain for $T_1z = z$

$$||z-T_2z||^2 \le (1-h)^2\,||z-T_2z||^2 + h^2q\;\max\{0,$$

$$0,$$

$$\left.\frac{||z-T_2z||^2}{1+||z-T_2z||^2}, 0, \frac{||z-T_2z||^2}{1+||z-T_2z||^2}\right\}$$

$$+2h^2(1-h)^2||z-T_2z||^2+0$$

$$\Rightarrow ||z-T_2z||^2 \le\; (1-h)||z-T_2z||^2 +$$

$$hq\,\frac{||z-T_2z||^2}{1+||z-T_2z||^2} +\; 2h\,(1-h)||z-T_2z||^2$$

$$\Rightarrow h(2h-1)||z-T_2z||^2 \le \frac{hq||z-T_2z||^2}{1+||z-T_2z||^2}$$

$\Rightarrow (2h-1)||z-T_2z||^2 + (2h-1)||z-T_2z||^4$

$\leq q||z-T_2z||^2$

$\Rightarrow (2h-1)||z-T_2z||^4 \leq [q-(2h-1)]||z-T_2z||^2$

$\Rightarrow ||z-T_2z||^4 \leq -\left[1-\frac{q}{2h-1}\right]||z-T_2z||^2$

for $0 < h \neq \frac{1}{2} < 1$ we have $\frac{q}{2h-1} = P < 1$

Suppose $z \neq T_2z$. Let $||z-T_2z||^2 = \delta$,

we have $\delta < -1\,[1-P]$

which is a contradiction.

Hence $z = T_2z$

Similarly we can prove that if $T_2z = z$, then $T_1z = z$

i.e. z is the common fixed point of T_1 and T_2.

This completes the proof.

Note :

Here we conclude this chapter by indicating an open question for further research work.

(i) If T satisfies contractive definition 5.0.1 and the continuity of mapping T is removed, does T have a fixed point ?

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